# boat time

Suppose you are anchored near the shore of a channel in which there is steady current, and you are going to run your (motor)boat at constant throttle to a dock directly across the channel on the opposite shore. There are two ways one might steer the boat to the dock:

- *the crabbing method*: steer a steady course with the nose of the boat pointed somewhat upstream, so the boat maintains a fixed orientation and crabs in a straight line across the channel

- *the pointing method*: keep the nose of the boat pointed directly at the dock
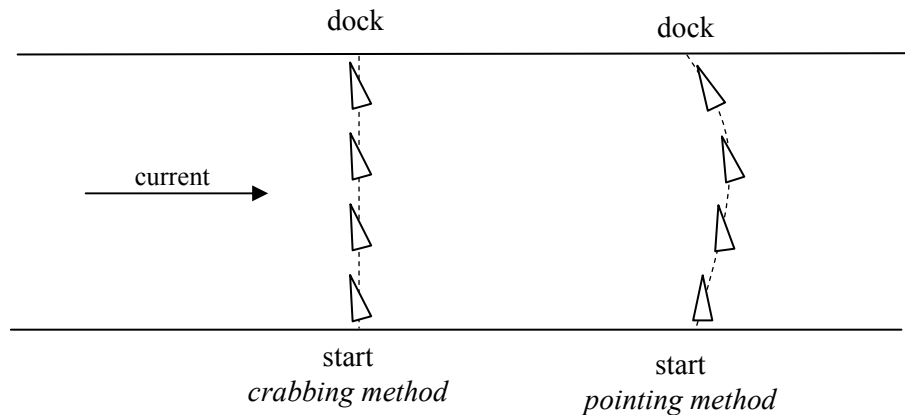
Which method gets the boat to the dock faster, and by how much? (Assume the boat runs at a constant speed relative to the water, which is faster than the speed of the current relative to the shore.)

Diego Scarabelli's Solution

The Crabbing Method

For the crabbing method we have the following situation,
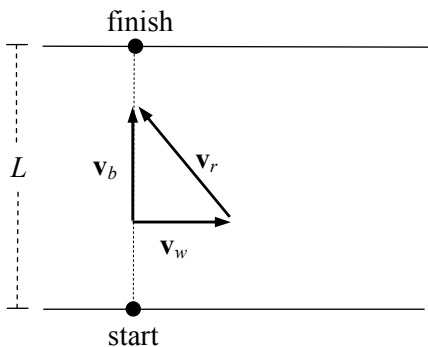
where $\mathbf{v}_b$ is the velocity of the boat, $\mathbf{v}_w$ is the velocity of the water, $\mathbf{v}_r$ is the velocity of the boat relative to the water, and L is the distance between the shores.

Thus, $v_b = \sqrt{v_r^2 - v_w^2}$, so the time it takes the boat to cross the channel using the crabbing method is

$$T_c = \frac{L}{v_b} = \frac{L}{\sqrt{v_r^2 - v_w^2}}. \qquad (1)$$

The Pointing Method

For the pointing method it is more convenient to consider the water to be still, and the finish point to be moving at constant speed in the direction opposite the current. We use a coordinate system with the start point as origin (note: positive x is *leftward*):
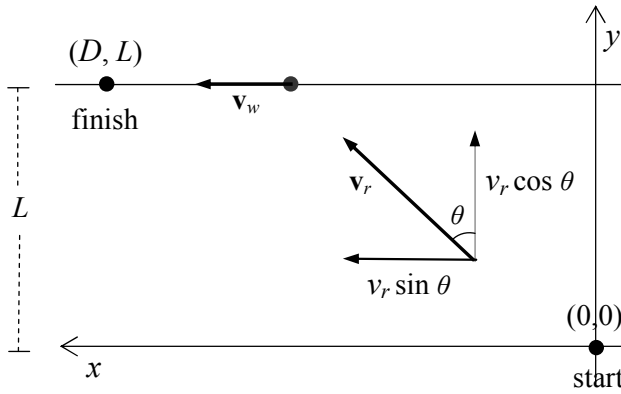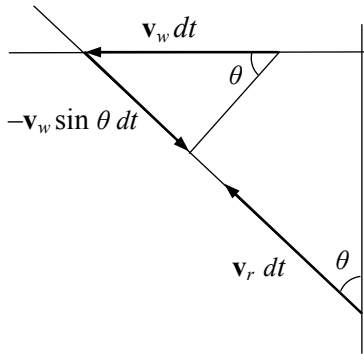
Here, $\mathbf{v}_w$ is the velocity of the finish point, $\mathbf{v}_r$ is the velocity of the boat, θ is the angle between $\mathbf{v}_r$ and the y-axis, and (D, L) are the coordinates of the finish point and the boat when they meet, after an interval of time T_p.

The x-displacement of the boat during an interval dt equals $dx = v_r\sin\theta\, dt$, so

$$D = \int_0^T v_r\sin\theta\, dt \tag{2}$$

And the finish point moves at constant speed, so

$$D = \int_0^T v_w\, dt = v_w T_p \tag{3}$$

The decrease in the distance between the boat and the finish point during the interval dt is $v_r\, dt - v_w\sin\theta\, dt$. Since this distance is initially equal to L,

$$L = \int_0^T v_r - v_w\sin\theta\, dt.$$

Setting $k = v_r/v_w$, we can rewrite this integral as

$$L = \int_0^T kv_w - \frac{v_r}{k}\sin\theta\, dt$$
$$= k\int_0^T v_w\, dt \;-\; \frac{1}{k}\int_0^T v_r\sin\theta\, dt. \tag{4}$$

Substituting eqs. (2) and (3) into (4),

$$L = kD - \frac{D}{k} = v_w T_p\left(k - \frac{1}{k}\right) = v_w T_p\left(\frac{v_r}{v_w} - \frac{v_w}{v_r}\right) = T_p\left(\frac{v_r^2 - v_w^2}{v_r}\right),$$

so $T_p = \dfrac{Lv_r}{v_r^2 - v_w^2}$. $\tag{5}$

From eqs. (1) and (5), $T_c/T_p = \sqrt{1 - \left(v_w/v_r\right)^2} < 1$. We therefore conclude that the crabbing method is faster than the pointing method.