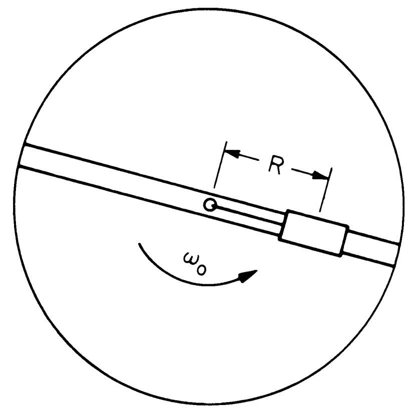
turntable cart

A turntable of moment of inertia I_0 rotates freely on a hollow vertical axis. A cart of mass m runs without friction on a straight radial track on the turntable. A cord attached to the cart passes over a small pulley and then downward through the hollow axis. Initially the entire system is rotating at angular speed ω_0, and the cart is at a fixed radius R from the axis. The cart is then pulled inward by applying an excess force to the cord, and eventually arrives at radius r, where it is allowed to remain.



a) What is the new angular velocity of the system?

b) Show in detail that the difference in the energy of the system between the two conditions is equal to the work done by the centripetal force.

c) If the cord is released, with what radial speed, *dr/dt*, will the cart pass the radius R?

Michael A. Gottlieb's Solution

a) The initial angular momentum of the system is $\left(I_0 + mR^2\right)\omega_0$, and the final angular momentum is $\left(I_0 + mr^2\right)\omega$, where ω is the final angular velocity. All forces on the cart are directed along a radius of the turntable, so no torques are exerted on the system and by conservation of angular momentum, the angular momentum remains constant, thus

$$\omega = \frac{\left(I_0 + mR^2\right)}{\left(I_0 + mr^2\right)}\,\omega_0.$$

b) The initial energy of the system is $\frac{1}{2}\left(I_0 + mR^2\right)\omega_0^2$, and the final energy of the system is $\frac{1}{2}(I_0 + mr^2)\omega^2$, which by (**a**) equals

$$\frac{(I_0 + mr^2)}{2}\cdot\left(\frac{I_0 + mR^2}{I_0 + mr^2}\,\omega_0\right)^2 = \frac{\left(I_0 + mR^2\right)^2}{2\left(I_0 + mr^2\right)}\,\omega_0^2.$$

Thus the difference in energy ΔE between the initial and final states is

$$\Delta E = \frac{\left(I_0 + mR^2\right)^2}{2\left(I_0 + mr^2\right)}\ \omega_0^2 - \frac{\left(I_0 + mR^2\right)}{2}\omega_0^2 \ =$$

$$\frac{1}{2}m\left(R^2 - r^2\right)\frac{\left(I_0 + mR^2\right)}{\left(I_0 + mr^2\right)}\omega_0^2.$$

Call the distance from the axis to the cart s, and call the tangential velocity of the cart at s, v_t. Then the centripetal force on the cart at s is mv_t^2/s, so the work done in pulling the cart from s=R to s=r is

$$W = \int_R^r mv_t^2/s\ ds\,.$$

Note that the angular velocity of the cart at s is $\omega_s = v_t/s$, so

$$W = \int_R^r m\omega_s^2 s\ ds\,.$$

By (**a**), $\omega_s = ((I_0 + mR^2)/(I_0 + ms^2))\ \omega_0$, so

$$W = \int_R^r m\left(\frac{I_0 + mR^2}{I_0 + ms^2}\ \omega_0\right)^2 s\ ds =$$

$$m\left(I_0 + mR^2\right)^2 \omega_0^2 \int_R^r \frac{s}{\left(I_0 + ms^2\right)^2}\ ds =$$

$$m\left(I_0 + mR^2\right)^2 \omega_0^2 \left[-\frac{1}{2m\left(I_0 + ms^2\right)}\right]_R^r =$$

$$\frac{1}{2}m\left(R^2 - r^2\right)\frac{\left(I_0 + mR^2\right)}{\left(I_0 + mr^2\right)}\omega_0^2$$

Thus, $W = \Delta E$.

c) Originally the cart had no radial velocity, it's (tangential) velocity was $\omega_0 R$ so its energy was

$$E_0 = \frac{1}{2} m (\omega_0 R)^2 .$$

When the cart returns to s = R after the cord is released, the system must have the same angular momentum it did originally (by conservation of angular momentum), so the tangential velocity of the cart is (again) $\omega_0 R$. If the radial velocity is now u, then the square of the velocity of the cart is $v^2 = u^2 + (\omega_0 R)^2$, so the total energy of the cart is

$$E = \frac{1}{2} m \left(u^2 + (\omega_0 R)^2 \right).$$

The cart has gained an additional energy $\frac{1}{2} mu^2$ which must come entirely from the work we did on it. So $W = \frac{1}{2} mu^2$. We find u by substitution from (**b**):

$$\frac{1}{2} m \left(R^2 - r^2 \right) \frac{\left(I_0 + mR^2 \right)}{\left(I_0 + mr^2 \right)} \omega_0^2 = \frac{1}{2} mu^2,$$

$$\therefore u = \omega_0 \sqrt{\left(R^2 - r^2 \right) \frac{\left(I_0 + mR^2 \right)}{\left(I_0 + mr^2 \right)}}.$$