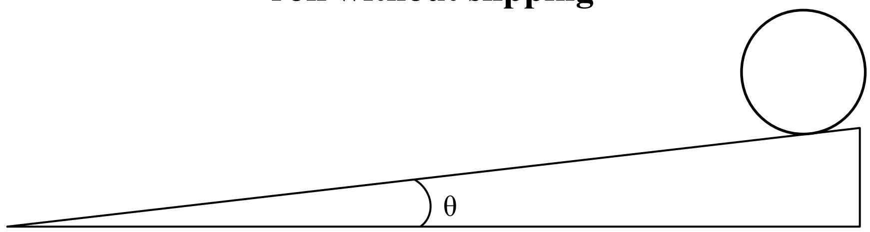
roll without slipping



A uniform solid ball is placed at rest on an incline of slope angle θ. What is the minimum value μ_0 of the coefficient of static friction between ball and incline so that the ball will roll down the incline without slipping?

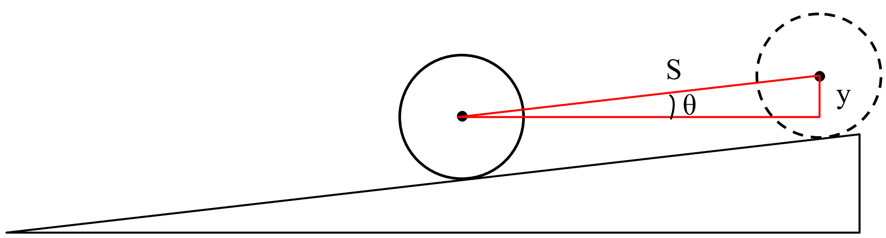
Solution by Michael A. Gottlieb

Let:

μ = coefficient of friction between ball and incline

M = mass of ball
R = radius of ball
I = moment of inertia of ball

S = displacement of ball's CM since it was at rest
y = the vertical distance the ball has descended since it was at rest



v = velocity of ball's CM
a = acceleration of ball's CM
ω = angular velocity of ball around its rolling axis

1. When a ball rolls without slipping and its CM has velocity v, the tangential velocity of the ball at its radius, R, normal to the axis of rotation, must also be v, and therefore $\omega = v/R$.

2. The angular momentum of the ball is (by definition) $I\omega$, and

$$I\omega = I\,(v/R) = (I/R)v.$$

3. The total torque on the ball when it rolls without slipping is the rate of change of its angular momentum, which equals

$$\frac{d(I\omega)}{dt} = \frac{d((I/R)v)}{dt} = (I/R)a\,.$$

4. The component of the ball's weight normal to the inclined plane is $Mg\cos\theta$. Thus, the frictional force on the ball is $\mu Mg\cos\theta$.

5. The frictional force is normal to the radius of the ball's axis of rotation (when it rolls) and it is applied at the surface of the ball (a distance R from the axis), so friction makes a torque on the ball equal to $R\mu Mg\cos\theta$.

6. In order for the ball not to slip, the torque on the ball from friction can not be less than the total torque on the ball when it rolls, and therefore $R\mu_0 Mg\cos\theta \geq (I/R)a$, or

$$\mu_0 \geq k\frac{a}{g\cos\theta},$$

with $k = I/MR^2$.

7. The total kinetic energy of the rolling ball equals

$$(1/2)Mv^2 + (1/2)I\omega^2 = (1/2)Mv^2 + (1/2)(kMR^2)(v/R)^2$$

$$= (1/2)\,(k+1)\,Mv^2.$$

Since energy is conserved (when the ball doesn't slip), the ball's total kinetic energy must equal the change in its potential energy, Mgy. Thus $(1/2)(k+1)Mv^2 = Mgy$ or $v^2 = 2gy/(k+1)$. Noting that $y = S\sin\theta$, we have

$$v^2 = 2gS\sin\theta/(k+1)$$

8. Taking the derivative with respect to time on both sides of (7), and substituting v = dS/dt and a = dv/dt, we have

$$2va = 2gv\sin\theta/(k+1)$$

or

$$a = g\sin\theta/(k+1).$$

9. Substituting (8) into (6), we have

$$\mu_0 \geq k\frac{g\sin\theta/(k+1)}{g\cos\theta}.$$

And therefore, in order for the ball to roll down the inclined plane without slipping,

$$\mu_0 \geq \frac{k}{k+1}\tan\theta = \frac{1}{1+(MR^2/I)}\tan\theta.$$

10. For a uniform ball the moment of inertia $I = (2/5)MR^2$ and therefore

$$\mu_0 \geq \frac{2}{7}\tan\theta$$