A particle starts from rest at the top of a frictionless sphere of radius $R$ and slides on the sphere under the force of gravity. How far below its starting point does it get before flying off the sphere?

**Solution by Gert Hamacher**

Let the distance in question be $h$, the mass of the particle be $m$, the centripetal acceleration of the particle be $a_c$, and the velocity of the particle be $v$. With these definitions the centripetal force is $ma_c$ and the gravitational force is $mg$.

During the slide down to the point where the particle flies off the sphere, there is a normal force on the particle from the sphere whose magnitude equals the radial component of the gravitational force minus the centripetal force, $mg \cos \theta - ma_c$. The normal force gets smaller and smaller as the particle speeds up, and at the point where the particle leaves the sphere, the normal force is zero, so that

$$mg \cos \theta - ma_c = 0,$$

$$\cos \theta = \frac{a_c}{g}.$$

Since with no friction the total energy is conserved,

$$\Delta E_{\text{total}} = \Delta E_{\text{kinetic}} + \Delta E_{\text{potential}} = \frac{1}{2}m \cdot v^2 - m \cdot g \cdot h = 0,$$

$$v^2 = 2g \cdot h,$$

and since $a_c = v^2/R$,

$$\cos \theta = \frac{a_c}{g} = \frac{v^2}{R \cdot g} = \frac{2g \cdot h}{R \cdot g} = \frac{2h}{R}.$$

By the geometry of the situation

$$\cos \theta = \frac{R-h}{R},$$

so

$$\frac{R-h}{R} = \frac{2h}{R},$$

and therefore

$$h = \frac{R}{3}.$$

We see that the solution is independent of the mass of the particle and of the acceleration due to gravity. So the solution is true for any particle on any planet ☺.