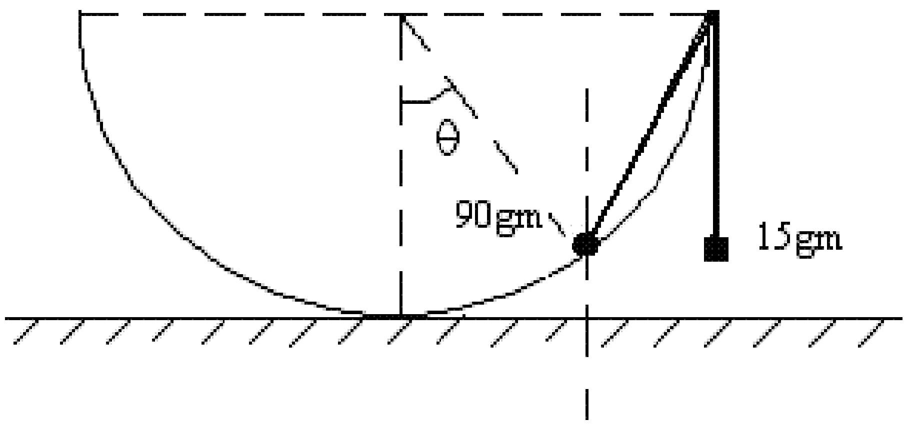
particle in bowl

This problem was contributed to the Feynman Lectures Website by Sukumar Chandra.



A hemispherical smooth bowl is held fixed with its axis vertical. A particle of mass 90gm. attached to one end of a light inextensible string is placed inside the bowl. The string passes over the frictionless rim of the bowl and is attached to another 15gm. mass hanging freely in air at its other end. In equilibrium position of the system of particles find the angle θ subtended by the radius through the 90gm. particle with the vertical.

Michael Gottlieb's Solution *(notes)*

The potential energy of a system in equilibrium is an extremum. To solve this problem, we will find the potential energy as a function of θ, differentiate it with respect to θ, set that equal to zero and solve for θ.

If h_{15} is the height of the 15gm. mass, and h_{90} is the height of the 90gm. mass (above some arbitrary datum level), the potential energy of the system is

$$\text{P.E.} = .015gh_{15} + .090gh_{90}.$$

Thus we must solve

$$\frac{d\text{P.E.}}{d\theta} = .015g\frac{dh_{15}}{d\theta} + .090g\frac{dh_{90}}{d\theta} = 0,$$

which is the same as

$$\frac{dh_{15}}{d\theta} + 6\frac{dh_{90}}{d\theta} = 0. \qquad (1)$$

In Fig. 1, where R is the radius of the bowl and L is the length of string extending from the 90gm. mass to the rim of the bowl, we see that

$$\sin\left(\left(\pi/2-\theta\right)/2\right)=\left(L/2\right)/R\,.$$

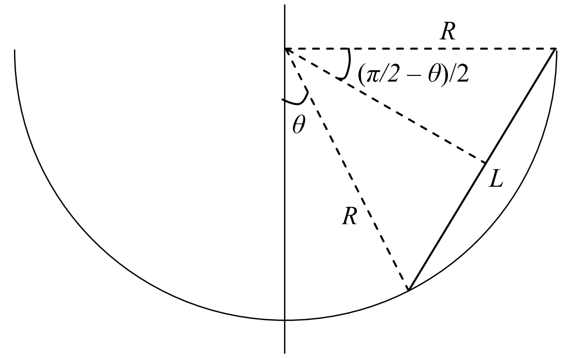


Figure 1

Noting that $dL = dh_{15}$, we find

$$\frac{dh_{15}}{d\theta}=\frac{dL}{d\theta}=-R\cos\left(\pi/4-\theta/2\right). \qquad (2)$$

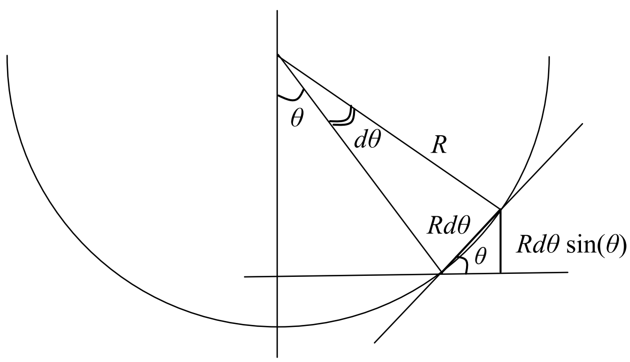


Figure 2

For the 90gm. mass on the spherical surface of the bowl, a small change in angle $d\theta$ results in a change of height equal to $R\,d\theta\sin\left(\theta\right)$ (see Fig. 2), so

$$\frac{dh_{90}}{d\theta}=R\sin\left(\theta\right). \qquad (3)$$

Substituting Eqs. (2) and (3) into (1), and simplifying, we get

$$6\sqrt{2}\sin\left(\theta\right)=\cos\left(\theta/2\right)+\sin\left(\theta/2\right).$$

Squaring both sides, and simplifying gives

$$72\sin^2\left(\theta\right)=1+\sin\left(\theta\right),$$

and solving the quadratic for positive sin θ, we arrive at the result:

$$\theta=\arcsin\frac{1}{8}.$$