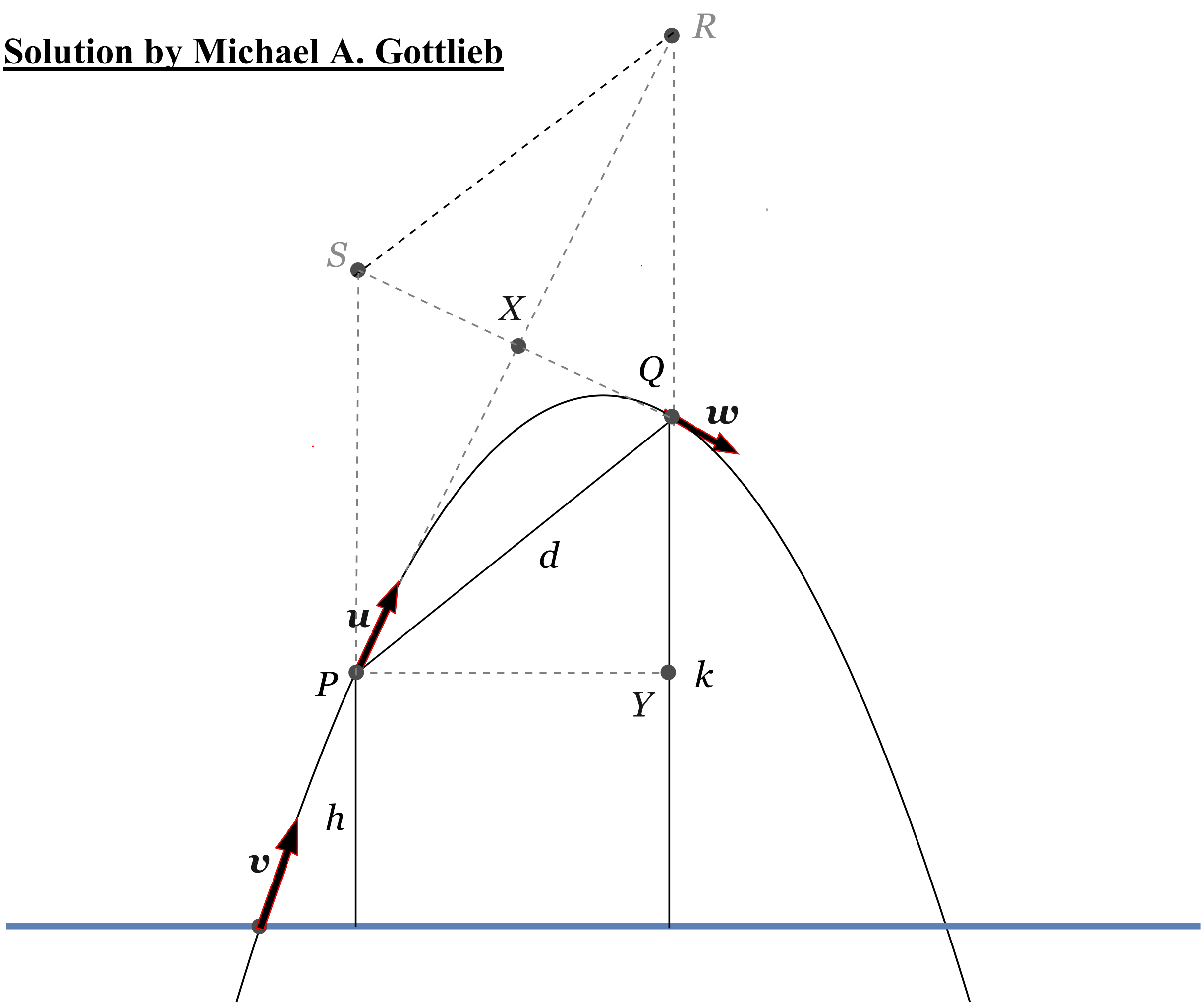
particle points parabola

P and Q are two points a distance d apart at heights h and k above a given horizontal plane. What is the minimum speed v with which a particle can be projected from the horizontal plane so as to pass through P and Q?

<u>Solution by Michael A. Gottlieb</u>



Let $\boldsymbol{u}$ be the velocity of the particle at P, let $\boldsymbol{w}$ be the velocity of the particle at Q, and consider the case $h = 0$, for which $\boldsymbol{v} = \boldsymbol{u}$, and the vertical distance between P and Q equals k.

Define a coordinate system with origin at point X, as shown in the figure, having one axis colinear with $\boldsymbol{u}$ – the "u-axis." For $u = |\boldsymbol{u}|$ to be the minimum speed required for the particle to get from point P to Q, it must be just fast enough so that the u-component of the particle's velocity equals zero when it arrives at Q. Thus $\boldsymbol{w}$ is perpendicular to $\boldsymbol{u}$, and the other axis of our coordinate system is colinear with $\boldsymbol{w}$ – the "w-axis."

Define point *R* on the *u*-axis directly above *Q*. If another particle were released from *R* at the same moment our particle departed *P* with velocity $\boldsymbol{u}$, our particle's velocity would always be pointing at it, since they would both accelerate downward at the same rate under the force of gravity; thus they would collide at *Q*[1]. If the duration of this process were t, then *RQ* would equal $gt^2/2$, where $\boldsymbol{g}$ is the acceleration of gravity. Now consider the *time-reversed* process, having equal duration *t*, with point *S* defined on the *w*-axis directly above *P*; then if another particle were released from *S* at the same moment our particle departed *Q* with velocity $-\boldsymbol{w}$, they would collide at *P* for the same reasons. Thus *SP* must also equal $gt^2/2$.

Since *RQ* and *SP* are parallel and equal in length, *PQRS* must be a parallelogram, and since its diagonals *PR* and *QS* are perpendicular, it must be a rhombus, with all sides of length *PQ* = d. In particular note that *PR* = 2*PX,* and that *RY* = *RQ* + *QY* = $d + k$.

The *u*-component of *PQ* is *PX,* and for the *u*-component of the particle's velocity to be zero when it reaches *Q,* kinematics dictates that

$$u^2 = 2g_u PX,$$

where g_u is the *u*-component of $\boldsymbol{g}$. Observing that $g_u = g \sin \measuredangle RPY = g\, RY/PR = g(d+k)/2PX$, we find that

$$u^2 = 2\,[g(d+k)/2PX]\,PX = g(d+k).$$

When $h > 0$, the vertical distance between *P* and *Q* equals $k - h$ instead of k, so the equation above becomes $u^2 = g(d + k - h)$, and conservation of energy dictates that $v^2 = u^2 + 2gh$, from which we conclude,

$$v^2 = g(d+k-h) + 2gh = g(d+k+h).$$

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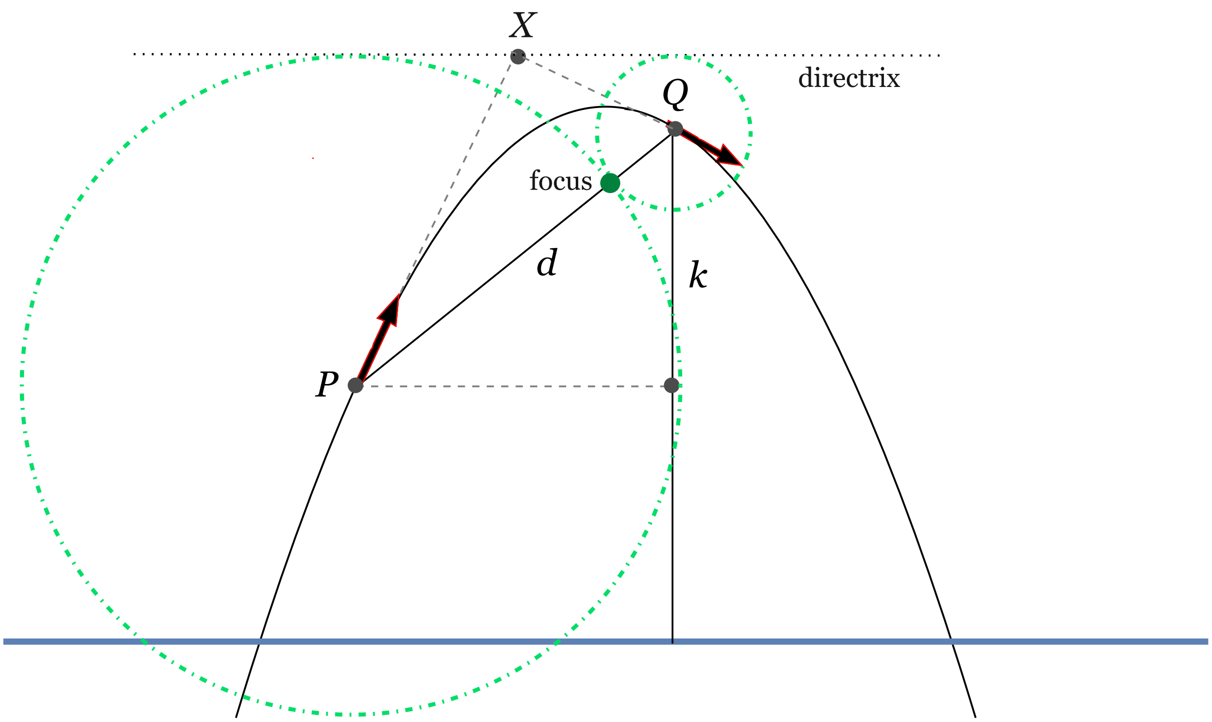
[1]This follows the famous "monkey hunter" experiment, as demonstrated by Caltech's inimitable *Feynman Lectures Hall Manager* Zach Tobin (youtu.be/yNKkbg2QZfE) and discussed in *Wikipedia* (en.wikipedia.org/wiki/Monkey_and_hunter).

Further Comments



A parabola can be defined as the locus of points equidistant from a fixed point (the focus) and a straight line (the directrix). It can be shown that perpendicular tangents to a parabola meet on its directrix, while the chord connecting the tangent points includes the focus. Thus the origin *X* of our *u*-*w* coordinate system lies on the directrix of the particle's parabolic path from *P* to *Q,* whose focus lies on the chord *PQ*. As a matter of interest I state without proof that the distance from *P* to the focus/directrix (the radius of the larger circle shown above) equals $(d + k)/2$, while the distance from *Q* to the focus/directrix (the radius of the smaller circle) equals $(d - k)/2$.