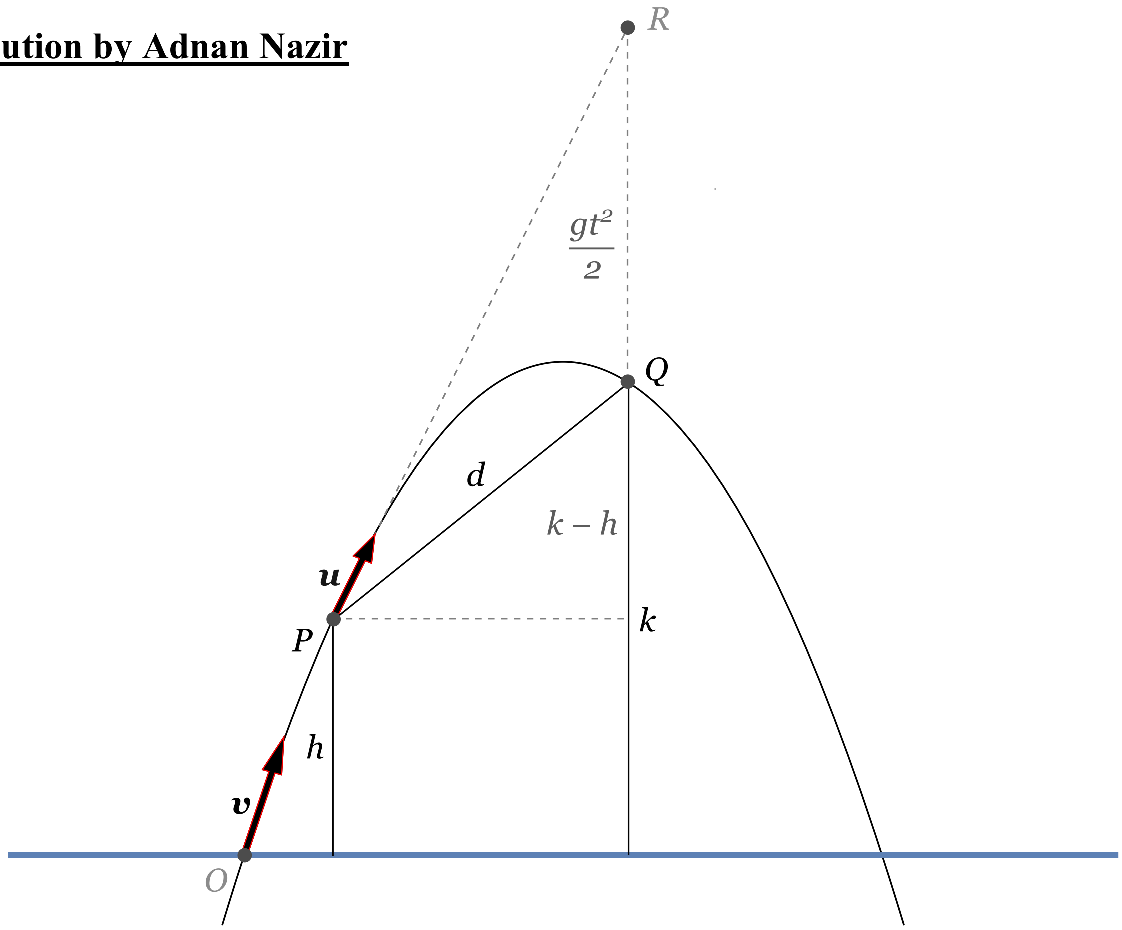
particle points parabola

P and Q are two points a distance d apart at heights h and k above a given horizontal plane. What is the minimum speed v with which a particle can be projected from the horizontal plane so as to pass through P and Q?

<u>Solution by Adnan Nazir</u>



The particle passes through the horizontal plane at point O with velocity $\boldsymbol{v}$. As it passes through point P at height h let its velocity be $\boldsymbol{u}$. By conservation of energy $\frac{1}{2}mv^2 = \frac{1}{2}mu^2 + mgh$ (for a particle with mass m), so

$$v^2 = u^2 + 2gh. \tag{1}$$

Thus, to minimize v we minimize u, or equivalently u^2.

Let t be the time it takes for the particle to go from point P to point Q at height k (assuming $k \geq h$, without loss of generality. Then by kinematics the horizontal and vertical components of u are respectively

$$u_x = \frac{\sqrt{d^2 - (k-h)^2}}{t},$$
$$u_y = \frac{\frac{1}{2}gt^2 + (k-h)}{t}.$$

Thus,

$$\begin{aligned} u^2 &= u_x^2 + u_y^2 \\ &= \frac{d^2 - (k-h)^2}{t^2} + \frac{\left(\frac{1}{2}gt^2 - (k-h)\right)^2}{t^2} \\ &= \left(\frac{d}{t} - \frac{gt}{2}\right)^2 + \ g(d+k-h). \end{aligned}$$

which is minimized when the first term equals 0, so that

$$u_{min}^2 = g(d+k-h). \quad (2)$$

Substituting the right side of Eq. (2) for u^2 in Eq. (1) gives

$$\begin{aligned} v_{min}^2 &= g(d+k-h) + 2gh \\ &= g(d+h+k). \end{aligned}$$

Therefore,

$$v_{min} = \sqrt{g(d+h+k)}.$$