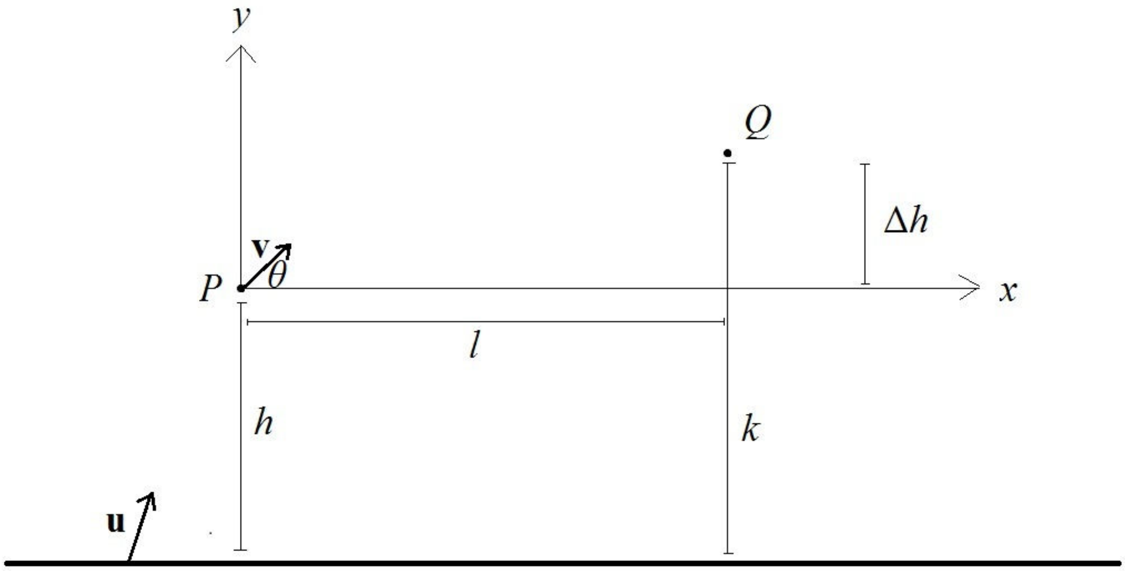
Particle points parabola

P and Q are two points a distance d apart at heights h and k above a given horizontal plane. What is the minimum speed v with which a particle can be projected from the horizontal plane so as to pass through P and Q?

Solution by Maciej Jarocki



Let us denote the initial speed of the particle as u and the speed at point P as v. As point P is at height h above the plane, the relation between the speeds is

$$\frac{1}{2}m\left(v^2 - u^2\right) + mgh = 0 \qquad (1)$$

by conservation of energy. Therefore, our task of minimizing the initial speed u is equivalent to minimizing the speed v at point P so that the projectile still passes through point Q.

A free projectile in the gravitational field obeys the equations

$$x = v\cos\theta t \text{ and } y = v\sin\theta t - \frac{1}{2}gt^2.$$

Here θ is the angle that the speed of the particle at point P makes with the x-axis. From these we can find the trajectory in the x-y plane:

$$y(x) = -\frac{gx^2}{2v^2}\sec^2\theta + x\tan\theta.$$

Let us take point P as the origin of our system (see figure). Then the coordinates of point Q are $\left(\sqrt{d^2 - \Delta h^2}, \Delta h\right)$ where $\Delta h \equiv k - h$. Let us further denote the x-coordinate as $\sqrt{d^2 - \Delta h^2} \equiv l$. Plugging this into our expression for the trajectory we obtain the constraint for v and θ:

$$\Delta h = -\frac{gl^2}{2v^2}\sec^2\theta + l\tan\theta. \tag{2}$$

The speed v must satisfy the above constraint for a given angle θ, therefore in fact it is a function of the angle: $v = v(\theta)$. Let us now take the derivative with respect to θ of both sides:

$$0 = -\frac{gl^2}{v^2}\sec^2\theta\left(\tan\theta - \frac{1}{v}\frac{\mathrm{d}v}{\mathrm{d}\theta}\right) + l\sec^2\theta,$$

but as the minimization condition is $\mathrm{d}v/\mathrm{d}\theta = 0$, this simplifies to

$$\tan\theta = \frac{v^2}{gl}$$

for the optimal angle θ. Using the identity $\sec^2\theta = 1 + \tan^2\theta$ and equation (2), we find

$$\Delta h = \frac{v^2}{2g} - \frac{gl^2}{2v^2},$$

and finally

$$v^2 = g\left(\Delta h + \sqrt{\Delta h^2 + l^2}\right) = g(\Delta h + d).$$

Now using this last result and equation (1), we obtain the final answer

$$u^2 = g(h + k + d).$$