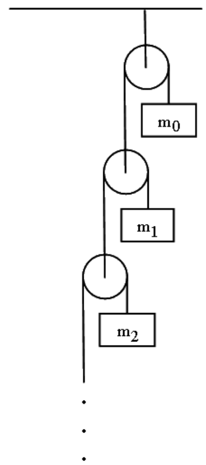
Infinite pulleys

An infinite series of pulleys and masses is arranged as shown, with $m_0 = 1/(1-t)$, and $m_i = t^{i-1}$ for $i > 0$, with $0 < t < 1$. At the moment the pulleys are released from rest, what is the acceleration a of mass m_0?



1 Solution by Fabrizio Gangemi

We solve the problem for a finite number of masses 0 through n, and then let $n \to \infty$.
If a_i denotes the acceleration of mass m_i, and T_i the tension of the string connected to it, the equation of motion for mass m_i is

$$T_i - m_i g = m_i a_i \ . \tag{1}$$

The force acting upon pulley i (for $i > 0$) is $T_{i-1} - 2T_i$. If $a_{p,i}$ is the acceleration of pulley i and $m_{p,i}$ is its mass, the equation of motion $T_{i-1} - 2T_i = m_{p,i} a_{p,i}$, with the assumption $m_{p,i} = 0$, implies $T_{i-1} = 2T_i$. Hence the tensions can be expressed as

$$\begin{aligned} T_i &= \frac{T_0}{2^i} \ , \qquad i = 0, \ldots, n-1 \ , \\ T_n &= \frac{T_0}{2^{n-1}} \ . \end{aligned}$$

For mass n the tension is the same as for mass $n-1$ because they share the same string.
The accelerations of masses and pulleys are constrained by the fact that each string is inextensible:

$$\begin{aligned} a_0 + a_{p,1} &= 0 \ , \\ a_1 - a_{p,1} + a_{p,2} - a_{p,1} &= 0 \ , \\ &\cdots \\ a_{n-1} - a_{p,n-1} + a_n - a_{p,n-1} &= 0 \ . \end{aligned}$$

Rearranging the terms, one has

$$\begin{aligned} a_{p,1} &= -a_0 \ , \\ a_{p,2} &= 2a_{p,1} - a_1 \ , \\ &\dots \\ a_n &= 2a_{p,n-1} - a_{n-1} \ . \end{aligned}$$

By substituting $a_{p,i}$ from each equation into the next one, the nth acceleration can be obtained as

$$a_n = -(2^{n-1}a_0 + 2^{n-2}a_1 + \ldots + a_{n-1}) = -2^{n-1}\sum_{i=0}^{n-1}\frac{a_i}{2^i} \ . \tag{2}$$

We may now rewrite the equations of motion 1 in the following form, where each term is divided by g, and the notations $\tau = T_0/g$, $\alpha_i = a_i/g$ are introduced:

$$\frac{\tau}{2^i m_i} = 1 + \alpha_i \qquad i = 0, \ldots, n-1 \ , \tag{3}$$

$$\frac{\tau}{2^{n-1} m_n} = 1 - 2^{n-1}\sum_{i=0}^{n-1}\frac{\alpha_i}{2^i} \ . \tag{4}$$

To take advantage of equation 2, we now multiply equation 3 by 2^{n-1-i} and sum over $i = 0, \ldots, n-1$ and then we add the result to equation 4, thus obtaining an equation for τ:

$$\tau\left(\frac{1}{2^{n-1}m_n} + \sum_{i=0}^{n-1}\frac{2^{n-i-1}}{2^i m_i}\right) = 1 + \sum_{i=0}^{n-1}2^{n-i-1} \ .$$

At this point we use the prescription for the masses, $m_i = t^{i-1}$, $i = 1, \ldots, n$, to obtain

$$\tau\left(\frac{1}{(2t)^{n-1}} + \frac{2^{n-1}}{m_0} + 2^{n-1}t\sum_{i=1}^{n-1}\frac{1}{(4t)^i}\right) = 2^n \ .$$

Finally, after multiplying both sides by $m_0/2^{n-1}$, we find the following expression for the tension:

$$\tau = \frac{2m_0}{1 + m_0\left(\frac{1}{(4t)^{n-1}} + t\sum_{i=1}^{n-1}\frac{1}{(4t)^i}\right)} \ .$$

The acceleration of mass m_0, according to equation 3 with $i = 0$, is given by

$$\alpha_0 = \frac{\tau}{m_0} - 1 = \frac{2}{1 + m_0\left(\frac{1}{(4t)^{n-1}} + t\sum_{i=1}^{n-1}\frac{1}{(4t)^i}\right)} - 1 \ . \tag{5}$$

Now, to take the limit for $n \to \infty$, we have to distinguish between two cases:

- when $4t \le 1$, the denominator on the right-hand side of equation 5 diverges, and we have $\alpha_0 \to -1$;
- when $4t > 1$, we get

$$\alpha_0 \to \frac{2}{1 + \frac{t}{1-t}\left(\frac{1}{1-\frac{1}{4t}} - 1\right)} - 1 = \frac{(2t-1)^2}{4t^2 - 6t + 1} \ .$$

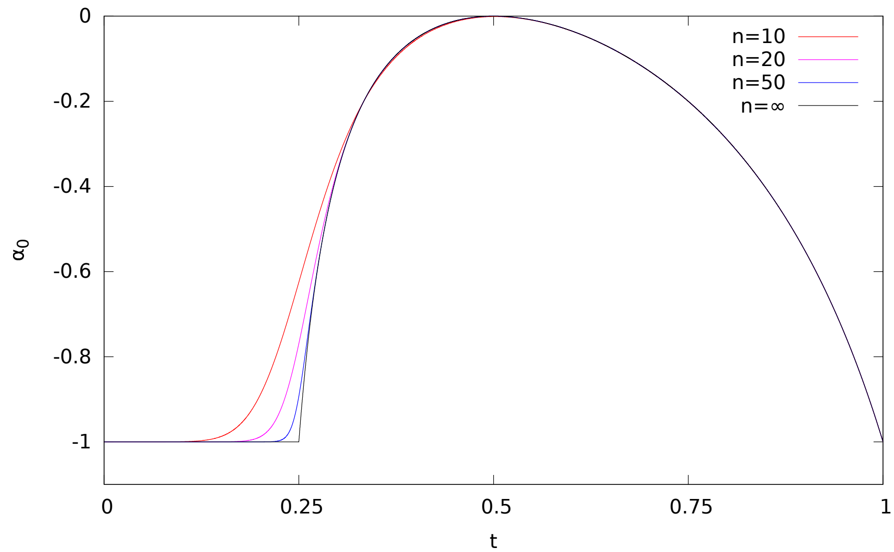


Figure 1: Plots of equation 5 for $n = 10, 20, 50$ and of equation 6 (black curve).

The denominator of the last expression can be written as $-4(t - t_-)(t_+ - t)$, with $t_\pm = (3 \pm \sqrt{5})/4$. Since $t_- < 1/4$ and $t_+ > 1$, there is no singularity in the range $(1/4, 1)$.
The solution may be summarized as

$$\alpha_0 = \begin{cases} -1 & 0 < t \leq \frac{1}{4} \\ \\ -\frac{(t-1/2)^2}{(t-t_-)(t_+-t)} & \frac{1}{4} < t < 1 \end{cases} \tag{6}$$

It is worth noting that α_0, as a function of t, is continuous at $t = 1/4$, but its derivative is not. The discontinuity of the derivative emerges after the limit $n \to \infty$ is taken: indeed, as can be seen by equation 5, α_0 is an analytic function of t in the whole range $(0, 1)$ for n finite. This is also shown in Figure 1, where equation 6 (black curve) is compared with equation 5 for some values of n.

2 Sign of the acceleration

The following argument may be used to determine the sign of the acceleration of mass m_0.
If we sum equation 1 over i and take into account that each tension is related to T_0 through $T_i = T_0/2^i$, we have

$$\sum_{i=0}^{\infty} m_i a_i = \sum_{i=0}^{\infty} \frac{T_0}{2^i} - \sum_{i=0}^{\infty} m_i g = 2T_0 - 2m_0 g \ ,$$

where the identity $\sum_{i>0} m_i = m_0$ has been used. Now, if we divide by $2m_0$, which is the mass of the whole system, we get the acceleration of the centre of mass:

$$a_{CM} = \frac{T_0}{m_0} - g \ .$$

This coincides with the acceleration a_0 of m_0 (see equation 1 for $i = 0$). Since the external forces are the total weight (downward) and the tension of the uppermost string (upward), which is a reaction force, a_{CM} cannot be upward, and the same holds for a_0. Therefore, we conclude $a_0 \leq 0$.